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The q-deformed supersymmetric t–J model with a boundary

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Abstract

The q-deformed supersymmetric t–J model on a semi-infinite lattice is diagonalized by using the level-one vertex operators of the quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$. We give the bosonization of the boundary states. We give an integral expression for the correlation functions of the boundary model, and derive the difference equations which they satisfy.

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1. Introduction

Integrable models with quantum superalgebra symmetries have been the focus of recent studies [1–6] in the context of strongly correlated fermion systems, a subject of high-profile international research activity because of their relevance to high-T_c superconductivity. The investigations of these models have largely been carried out within the framework of QISM and the Bethe ansatz method. The exceptions are references [7], where the algebraic analysis method, developed in [8, 9] and generalized in [11–15], was used to diagonalize the supersymmetric t–J model and its multi-component version directly on an infinite lattice.

The algebraic analysis method [8, 9], which we will call the vertex operator method, was formulated with the help of the level-one q-vertex operators [10] and highest-weight representations of quantum affine algebras. The vertex operator method was later extended in [16] to treat integrable models with boundary interactions [17, 18]. It was shown in [16] how the space of states of the boundary XXZ spin-$\frac{1}{2}$ chain on a semi-infinite lattice can be described in terms of level-one q-vertex operators of $U_q(\widehat{sl_2})$, and how the correlation functions can be computed using the vertex operators. Several other models have been analysed by means of this approach [19–22].

In this paper, we study the q-deformed supersymmetric t–J model with an integrable boundary. We will work directly on a semi-infinite lattice. As is known, the q-deformed supersymmetric t–J model on an infinite lattice (i.e. without a boundary) has as its symmetry algebra the quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$ [7]. On a finite lattice with diagonal boundary reflection K-matrices, this model was solved in [6] by the Bethe ansatz method. Here we adopt the vertex operator method. We will diagonalize the boundary model Hamiltonian directly on the semi-infinite lattice, and moreover compute the correlation functions of the boundary model.

This paper is organized as follows. In section 2, we describe the vertex operator approach to the q-deformed supersymmetric t–J model on the semi-infinite lattice. In section 3, we study the bosonic realization of the boundary states associated with the level-one highest-weight representation of $U_q[\widehat{sl(2|1)}]$. In section 4, we compute the correlation functions of the local operators (including the spin operator S_1^z) and derive the difference equations which they satisfy. In appendix A, we review the bosonization of $U_q[\widehat{sl(2|1)}]$ at level one and the associated vertex operators.

2. Boundary q-deformed supersymmetric t–J model

2.1. q-deformed supersymmetric t–J model on a finite lattice

In this section, we recall some facts about the q-deformed supersymmetric t–J model on a finite lattice. Throughout this paper, we fix q such that $|q| < 1$.

Let V be the three-dimensional graded vector space and E_{ij} be the 3×3 matrix whose (i, j) element is unity and whose elements are zero otherwise. The grading of the basis vectors v_1, v_2, v_3 of V is chosen to be $[v_1] = [v_2] = 1, [v_3] = 0$. Let V^* be the dual space and $\{v_1^*, v_0^*, v_{-1}^*\}$ the dual basis vectors. Denote by V_z (V_z^{*S}) the three-dimensional level-0 representation (dual representation) of $U_q[\widehat{sl(2|1)}]$ associated with V. Let $R(z) \in \mathrm{end}(V \otimes V)$ be the R-matrix of $U_q[\widehat{sl(2|1)}]$ with matrix elements defined by

$$R(z)(v_i \otimes v_j) = \sum_{k,l} R_{kl}^{ij}(z) v_k \otimes v_l, \qquad \forall v_i, v_j, v_k, v_l \in V,$$

where

$$\begin{aligned}
R_{33}^{33}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1q^{-1} - z_2q}{z_1q - z_2q^{-1}}, & R_{23}^{23}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}},\\
R_{23}^{32}\left(\frac{z_1}{z_2}\right) &= \frac{(q - q^{-1})z_2}{z_1q - z_2q^{-1}}, & R_{32}^{32}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}},\\
R_{32}^{23}\left(\frac{z_1}{z_2}\right) &= \frac{(q - q^{-1})z_1}{z_1q - z_2q^{-1}}, & R_{22}^{22}\left(\frac{z_1}{z_2}\right) &= -1,\\
R_{13}^{13}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}}, & R_{13}^{31}\left(\frac{z_1}{z_2}\right) &= \frac{(q - q^{-1})z_2}{z_1q - z_2q^{-1}},\\
R_{31}^{31}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}}, & R_{31}^{13}\left(\frac{z_1}{z_2}\right) &= \frac{(q - q^{-1})z_1}{z_1q - z_2q^{-1}},\\
R_{12}^{12}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}}, & R_{12}^{21}\left(\frac{z_1}{z_2}\right) &= -\frac{(q - q^{-1})z_2}{z_1q - z_2q^{-1}},\\
R_{21}^{21}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1q - z_2q^{-1}}, & R_{21}^{12}\left(\frac{z_1}{z_2}\right) &= -\frac{(q - q^{-1})z_1}{z_1q - z_2q^{-1}},\\
R_{11}^{11}\left(\frac{z_1}{z_2}\right) &= -1, & R_{kl}^{ij} &= 0 \qquad \text{otherwise.}
\end{aligned}$$

The R-matrix satisfies the graded Yang–Baxter equation (YBE) on $V \otimes V \otimes V$:

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),$$

and moreover obeys: (i) the initial condition $R(1) = P$, with P being the graded permutation operator; (ii) the unitarity condition $R_{12}(z/w)R_{21}(w/z) = 1$, where $R_{21}(z) = PR_{12}(z)P$; and (iii) cross-unitarity,

$$R^{-1,st_1}(z)((M \otimes 1)R(zq^{-2})(M \otimes 1))^{st_1} = 1 \otimes 1,$$

where

$$M \equiv q^{2\bar{\rho}} \stackrel{\text{def}}{=} \begin{pmatrix} q^{2\rho_1} & & \\ & q^{2\rho_2} & \\ & & q^{2\rho_3} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & q^{-2} & \\ & & q^{-2} \end{pmatrix}. \tag{2.1}$$

The various supertranspositions of the R-matrix are given by

$$(R^{st_1}(z))_{ij}^{kl} = R(z)_{kj}^{il}(-1)^{[i]([i]+[k])}, \qquad (R^{st_2}(z))_{ij}^{kl} = R(z)_{il}^{kj}(-1)^{[j]([j]+[l])},$$
$$(R^{st_{12}}(z))_{ij}^{kl} = R(z)_{kl}^{ij}(-1)^{([i]+[j])([i]+[j]+[l]+[k])} = R(z)_{kl}^{ij}.$$

Following Sklyanin [17], we construct the transfer matrix of an integrable finite chain, with an open boundary condition described by a reflection K-matrix $K(z)$. Here $K(z)$ is a solution of the graded reflection equation

$$K_2(z_2)R_{21}(z_1z_2)K_1(z_1)R_{12}(z_1/z_2) = R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1z_2)K_2(z_2). \tag{2.2}$$

With appropriate normalization, we can show that this $K(z)$ obeys the relations

$$\begin{aligned} &K(1) = 1, && \text{(boundary initial condition)}, \\ &K(z)K(z^{-1}) = 1, && \text{(boundary unitarity)}, \\ &\bar{K}(z)\bar{K}(z^{-1}) = 1, && \text{(boundary cross-unitarity)}, \end{aligned} \tag{2.3}$$

where $\bar{K}(z)$ is defined by

$$\bar{K}(z) = -\sum_{\alpha,\beta} R(z^2)_{i\beta}^{\alpha j}(-1)^{[i]+[j]+[j][\beta]+[\alpha][\beta]}K_{\alpha}^{\beta}(z^{-1}q^{-1})q^{2\rho_\alpha}. \tag{2.4}$$

The third relation is the graded extension of the boundary cross-unitarity [16, 18, 23].

The transfer matrix of the q-deformed supersymmetric t–J model on a finite chain with the open boundary condition is constructed from $R(z)$ and $K(z)$ via [17, 24]

$$T_B^{\text{fin}}(z) = \text{str}_{V_0}(K^+(z)\mathcal{T}(z^{-1})K(z)\mathcal{T}(z)), \tag{2.5}$$

where $K^+(z) = K(-z^{-1}q^{-3})^{st}M$ and

$$\mathcal{T}(z) = R_{01}(z)\cdots R_{0N}(z) \in \text{end}(V_0 \otimes V_1 \otimes \cdots \otimes V_N)$$

is the double-row monodromy matrix. The supertrace is defined as $\text{str}(A) = \sum(-1)^{[i]}A_{ii}$.

It can be verified that the $T_B^{\text{fin}}(z)$ form a commuting family; $[T_B^{\text{fin}}(z), T_B^{\text{fin}}(w)] = 0$. The Hamiltonian of the boundary q-deformed supersymmetric t–J model is given by [6, 17]

$$H_B^{\text{fin}} = \frac{\mathrm{d}}{\mathrm{d}z}T_B^{\text{fin}}(z)|_{z=1} = \sum_{j=1}^{N-1} h_{j,j+1} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}K(z)|_{z=1} + \frac{\text{str}_{V_0}(K^+(1)h_{0,N})}{K^+(1)}, \tag{2.6}$$

where $h_{j,j+1} = P_{j,j+1}(\mathrm{d}/\mathrm{d}z)R_{j,j+1}(z)|_{z=1}$.

The transfer matrix (2.5) with diagonal reflection K-matrices was diagonalized by the Bethe ansatz method in [6].

2.2. q-deformed supersymmetric t–J model on a semi-infinite lattice

In this paper, we restrict ourselves to the diagonal reflection K-matrix of the form

$$K(z) = f(z)\begin{pmatrix} [(1-rz)/(z-r)]z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad f(z) = \frac{\phi(z|r)}{\phi(z^{-1}|r)}, \qquad \phi(z|r) = \frac{1}{1-rz}, \tag{2.7}$$

where r is an arbitrary parameter which is related to the boundary interaction [16, 18]. One can check that such a K-matrix satisfies boundary unitarity and cross-unitarity (2.3).

We now consider Hamiltonian (2.6) in the semi-infinite limit:

$$H_B^{\rm fin}|_{N\longrightarrow\infty} = \sum_{j=1}^{\infty} h_{j,j+1} + \Delta, \tag{2.8}$$

where $\Delta = \frac{1}{2}(\mathrm{d}/\mathrm{d}z) f(z)|_{z=1}$ acts formally on the left-infinite tensor product space

$$\cdots \otimes V \otimes V. \tag{2.9}$$

As mentioned in the introduction, the q-deformed supersymmetric t–J model on an infinite lattice has $U_q[\widehat{sl(2|1)}]$ as its symmetry algebra. Let $V(\mu_\alpha)$ be the level-one irreducible highest-weight $U_q[\widehat{sl(2|1)}]$-modules with highest weight μ_α, $\alpha \in \mathbb{Z}$ (see (A.4) and [7]). Consider the level-one vertex operators which are intertwining operators for $V(\mu_\alpha)$ and $V(\mu_\beta)$. It has been shown in [7] that the following type I vertex operators $\Phi(z)$ exist: $\Phi^*(z)$ which intertwine the level-one irreducible highest-weight $U_q[\widehat{sl(2|1)}]$-modules $V(\mu_\alpha)$:

$$\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_z, \qquad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V_z^{*S}. \tag{2.10}$$

(See appendix A for more details on $V(\mu_\alpha)$ and its associated vertex operators.) Therefore, following [7, 8, 16, 21], we can write the transfer matrix of the q-deformed supersymmetric t–J model on the semi-infinite lattice as

$$T_B(z) = -\sum_{i,j=1}^{3} \Phi_i^*(z^{-1}) K_i^j(z) \Phi_j(z)(-1)^{[i]} = \sum_{i,j=1}^{3} q^{-2\rho_j} \Phi_j(z) \bar{K}_i^j(z^{-1}q^{-1}) \Phi_i^*(z^{-1}), \tag{2.11}$$

where $\Phi_i(z)$ and $\Phi_j^*(z)$ are the components of the $U_q[\widehat{sl(2|1)}]$ vertex operators of type I (see (A.6)). We have used the exchange relations of vertex operators (A.14) and the definition of $\bar{K}(z)$ (2.4) in the above equation.

We remark that the transfer matrix $T_B(z)$ given by (2.11) is an operator with the property

$$T(z) : V(\mu_\alpha) \longrightarrow V(\mu_\alpha), \qquad \alpha \in \mathbb{Z}.$$

The commutativity of the transfer matrix (2.11), $[T_B(z), T_B(w)] = 0$, then follows from (A.12) and (2.2). Moreover by (A.12), (A.15) and (A.16), one can show that

$$T_B(1) = \mathrm{id}, \qquad T_B(z)T_B(z^{-1}) = \mathrm{id}, \tag{2.12}$$

$$T_B(z)T_B(z^{-1}q^{-2}) = \mathrm{id}. \tag{2.13}$$

These relations correspond to the boundary initial condition, boundary unitarity and boundary cross-unitarity (2.3) of the K-matrix, respectively. In terms of the transfer matrix, the q-deformed supersymmetric t–J model Hamiltonian on the semi-infinite lattice is given by

$$H = \frac{\mathrm{d}}{\mathrm{d}z} T_B(z)|_{z=1}. \tag{2.14}$$

Following [7], we define the local operators acting on the nth site:

$$E^{(1)}_{i,j} = -\Phi^*_i(1)\Phi_j(1)(-1)^{[j]}, \tag{2.15}$$

$$E^{(n)}_{i,j} = \sum_m (-1)^{([i]+[j])[m]+[m]}\Phi^*_m(1)E^{(n-1)}_{i,j}\Phi_m(1), \qquad n = 2, 3, \ldots. \tag{2.16}$$

In particular, we have the spin operator S^z_1

$$S^z_1 = \tfrac{1}{2}(E^{(1)}_{11} - E^{(1)}_{22}) = \tfrac{1}{2}\{\Phi^*_1(1)\Phi_1(1) - \Phi^*_2(1)\Phi_2(1)\}.$$

3. The boundary states

In this section we construct the bosonic boundary state $|\alpha; r\rangle_B$ and its dual state ${}_B\langle r; \alpha|$, which satisfy

$$T_B(z)|\alpha; r\rangle_B = |\alpha; r\rangle_B, \qquad {}_B\langle r; \alpha|T_B(z) = {}_B\langle r; \alpha|. \tag{3.1}$$

By (A.15) and (2.11), the above eigenvalue problem is equivalent to

$$\Phi_i(z^{-1})|\alpha; r\rangle_B = \sum_j K^j_i(z)\Phi_j(z)|\alpha; r\rangle_B, \tag{3.2}$$

$${}_B\langle r; \alpha|\Phi^*_j(z)(-1)^{[j]} = \sum_i {}_B\langle r; \alpha|\Phi^*_i(z^{-1})K^j_i(z)(-1)^{[i]}. \tag{3.3}$$

3.1. The boundary state in $V(\Lambda_0)$

First, we consider the boundary state $|0; r\rangle_B \in V(\mu_0)$ (or $V(\Lambda_0)$). As is shown in appendix A, $V(\mu_0) = \eta_0\xi_0 F_{(0;\beta)}$ and the highest-weight vector $|\Lambda_0\rangle = |\beta, \beta, \beta, 0\rangle$ satisfies

$$\eta_0|\Lambda_0\rangle = 0.$$

So we make the following ansatz [16, 27]:

$$|0; r\rangle_B = e^{F_0(r)}|\Lambda_0\rangle, \tag{3.4}$$

$$F_0(r) = \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{[m]^2}\alpha_m\{h^1_{-m}h^{*1}_{-m} + h^2_{-m}h^{*2}_{-m} + c_{-m}c_{-m}\} + \sum_{m=1}^{\infty}\{\beta^1_m h^1_{-m} + \beta^2_m h^2_{-m} + \beta^3_m c_{-m}\}, \tag{3.5}$$

where $\alpha_m, \beta^1_m, \beta^2_m, \beta^3_m$ are functions of the boundary parameter r.

We can check that $e^{F_0(r)}$ plays a role of the Bogoliubov transformation:

$$e^{-F_0(r)}h^{*1}_m e^{F_0(r)} = h^{*1}_m + \alpha_m h^{*1}_{-m} + \frac{[m]^2}{m}\beta^1_m,$$

$$e^{-F_0(r)}h^{*2}_m e^{F_0(r)} = h^{*2}_m + \alpha_m h^{*2}_{-m} + \frac{[m]^2}{m}\beta^2_m,$$

$$e^{-F_0(r)}c_m e^{F_0(r)} = c_m + \alpha_m c_{-m} + \frac{[m]^2}{m}\beta^3_m,$$

$$e^{-F_0(r)}h^1_m e^{F_0(r)} = h^1_m + \alpha_m h^1_{-m} + \frac{[2m][m]}{m}\beta^1_m - \beta^2_m\frac{[m]^2}{m}.$$

Keeping (3.2) in mind and following [16, 19, 21], we find that the coefficients $\alpha_m, \beta^1_m, \beta^2_m, \beta^3_m$ are

$$\alpha_m = -q^{4m}, \qquad \beta_m^1 = 0, \tag{3.6}$$

$$\beta_m^2 = \frac{r^m}{[m]} q^{\frac{3}{2}m} + \theta_m \frac{q^{\frac{3}{2}m} - q^{\frac{5}{2}m}}{[m]}, \tag{3.7}$$

$$\beta_m^3 = \theta_m \frac{q^{2m}}{[m]}, \tag{3.8}$$

where the function θ_m is defined by

$$\theta_m = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Moreover following [19] one can check that $\eta_0|0\rangle_B = 0$, i.e. the boundary state $|0\rangle_B \in V(\mu_0)$, as required. In the derivation, the following relations are useful:

$$\begin{aligned}
&\mathrm{e}^{h_1^{*+}(\xi^{-1}q^2;-\frac{1}{2})}|0;r\rangle_B = \mathrm{e}^{h_1^{*-}(\xi q^2;-\frac{1}{2})}|0;r\rangle_B, \\
&\mathrm{e}^{-h_1^{+}(\omega q^2;-\frac{1}{2})}|0;r\rangle_B = (1-\omega^{-2})(1-r\omega^{-1})\mathrm{e}^{-h_1^{-}(\omega^{-1}q^2;-\frac{1}{2})}|0;r\rangle_B, \\
&\mathrm{e}^{c^{+}(\omega q^2;0)}|0\rangle_B = (1-\omega^{-2})\mathrm{e}^{c^{-}(\omega^{-1}q^2;0)}|0;r\rangle_B, \\
&\mathrm{e}^{-h_2^{*+}(\xi^{-1}q^2;-\frac{1}{2})}|0\rangle_B = (1-\omega r)^{-1}\mathrm{e}^{-h_2^{*-}(\xi q^2;-\frac{1}{2})}|0;r\rangle_B.
\end{aligned}$$

Similarly, the dual state ${}_B\langle r;0| \in V^*(\mu_0)$ can be constructed:

$${}_B\langle r;0| = \langle 0|\mathrm{e}^{G_0(r)}, \tag{3.9}$$

$$G_0(r) = -\frac{1}{2}\sum_{m=1}^{\infty} q^{-2m}\frac{m}{[m]^2}\{h_m^1 h_m^{1*} + h_m^2 h_m^{2*} + c_m c_m\} + \sum_{m=1}^{\infty}\{\delta_m^1 h_m^1 + \delta_m^2 h_m^2 + \delta_m^3 c_m\}, \tag{3.10}$$

where

$$\delta_m^1 = 0, \qquad \delta_m^2 = -\frac{r^{-m}q^{-\frac{m}{2}}}{[m]} + \theta_m\left(\frac{q^{-\frac{1}{2}m} + q^{-\frac{3}{2}m}}{[m]}\right), \qquad \delta_m^3 = \theta_m\left(\frac{q^{-m}}{[m]}\right).$$

3.2. *The general boundary states*

Noting that the boundary K-matrix $K(z)$ have the following properties:

$$K(z)|_{z=r} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{3.11}$$

we may define $|-1;r\rangle_B = \Phi_1(r^{-1})|0;r\rangle_B|_{r\longrightarrow rq^{-2}}$. One can check that such a $|-1;r\rangle_B$ satisfies (3.2) with $\alpha = -1$. Recursively, we can construct the general boundary state $|\alpha;r\rangle_B$ from $|0;r\rangle_B$ by means of the following recursive relations:

$$|\alpha;rq^2\rangle_B = \Phi_1(r^{-1})|\alpha+1;r\rangle_B, \qquad |\alpha;r\rangle_B = q^{-2\rho_1}\Phi_1^*(r^{-1}q^2)|\alpha-1;rq^2\rangle_B. \tag{3.12}$$

We have used the second invertibility relation (A.16). Similarly, we can obtain the dual boundary states ${}_B\langle r;\alpha|$ from ${}_B\langle r;0|$ by means of the recursive relations

$${}_B\langle r;\alpha|\Phi_1^*(r) = {}_B\langle rq^2;\alpha-1|, \qquad {}_B\langle r;\alpha| = {}_B\langle rq^2;\alpha-1|\Phi_1(rq^{-2})q^{-2\rho_1}. \tag{3.13}$$

4. Correlation functions

The aim of this section is to calculate the one-point functions $\langle E_{i,j}^{(1)}\rangle_\alpha$:

$$\langle E_{i,j}^{(1)}\rangle_\alpha = \frac{{}_B\langle r;\alpha|E_{i,j}^{(1)}|\alpha;r\rangle_B}{{}_B\langle r;\alpha|\alpha;r\rangle_B}.$$

The generalization to the calculation of multi-point functions is straightforward. Thanks to the recursive relations (3.12) and (3.13), it is sufficient to calculate $\langle E^{(1)}_{i,j}\rangle_0$. Thus in the following we restrict ourselves to the calculation of $\langle E^{(1)}_{i,j}\rangle_0$.

Define

$$\oint \mathrm{d}z\, f(z) = f_{-1}, \qquad \text{for formal series function } f(z) = \sum_{n\in\mathbb{Z}} f_n z^n.$$

From the bosonic realization of Drinfeld currents of $U_q[\widehat{sl(2|1)}]$, equations (A.7)–(A.10) and the normal ordering relations in the appendix A, we obtain the integral expressions for the vertex operators [7]:

$$\phi_3(z) = \,:\mathrm{e}^{-h_2^*(q^2z;-\frac{1}{2})+c(q^2z;0)}:\mathrm{e}^{-\mathrm{i}\pi a_0^2},$$

$$\phi_2(z) = \,:\left\{\frac{\mathrm{e}^{-c(wq;0)}}{wq(1-qz/w)} + \frac{\mathrm{e}^{-c(wq^{-1};0)}}{zq^2(1-w/zq^3)}\right\}\mathrm{e}^{-h_2^*(q^2z;-\frac{1}{2})-h_2(w;-\frac{1}{2})+c(q^2z;0)}\mathrm{e}^{\mathrm{i}\pi h_0^1}:,$$

$$\phi_1(z) = \frac{q^2-1}{w(1-w_1q/w)(1-wq/w_1)}:\left\{\frac{\mathrm{e}^{-c(wq;0)}}{wq(1-qz/w)} + \frac{\mathrm{e}^{-c(wq^{-1};0)}}{zq^2(1-w/zq^3)}\right\}$$
$$\times\, \mathrm{e}^{-h_2^*(q^2z;-\frac{1}{2})-h_2(w;-\frac{1}{2})-h_1(w_1;-\frac{1}{2})+c(q^2z;0)}\mathrm{e}^{-\mathrm{i}\pi a_0^2}:,$$

$$\phi_1^*(z) = \,:\mathrm{e}^{h_1^*(qz;-\frac{1}{2})}:\mathrm{e}^{\mathrm{i}\pi a_0^2},$$

$$\phi_2^*(z) = \oint \mathrm{d}w\, \frac{1-q^{-2}}{z(1-zq^2/w)(1-w/z)}:\mathrm{e}^{h_1^*(qz;-\frac{1}{2})-h_1(w;-\frac{1}{2})}\mathrm{e}^{-\mathrm{i}\pi h_0^1}:,$$

$$\phi_3^*(z) = \oint \mathrm{d}w_1 \oint \mathrm{d}w\, \frac{1-q^{-2}}{z(1-zq^2/w)(1-w/z)}$$
$$\times\,:\frac{\mathrm{e}^{-c(w_1q;0)}-\mathrm{e}^{-c(w_1q^{-1};0)}}{ww_1(1-wq/w_1)(1-w_1q/w)}\mathrm{e}^{h_1^*(qz;-\frac{1}{2})-h_1(w;-\frac{1}{2})-h_2(w_1;-\frac{1}{2})}\mathrm{e}^{\mathrm{i}\pi a_0^2}:.$$

Since $\eta_0|0,r\rangle = 0$, one may set

$$P_{i,j}(z_1,z_2) = \frac{{}_B\langle r;0|\Phi_i^*(z_1)\Phi_j(z_2)|0;r\rangle_B}{\langle r;0|0;r\rangle_B} \equiv \frac{{}_B\langle r;0|\phi_i^*(z_1)\phi_j(z_2)|0;r\rangle_B}{\langle r;0|0;r\rangle_B}; \tag{4.1}$$

then $\langle E^{(1)}_{i,j}\rangle_0 = -(-1)^{[j]}P_{i,j}(1,1)$.

The bosonization formulae (A.7)–(A.10) of the vertex operators immediately imply

$$P_{i,j}(z_1,z_2) = \delta_{ij}F_i(z_1,z_2) \stackrel{\text{def}}{=} \delta_{ij}\frac{{}_B\langle r;0|\phi_i^*(z_1)\phi_i(z_2)|0;r\rangle_B}{{}_B\langle r;0|0;r\rangle_B}.$$

Using the technique in [16, 21] (see equation (C.4)), after tedious calculation, we get

$$\begin{aligned}{}_B\langle r;0|0;r\rangle_B &= \prod_{n=1}^{\infty}\frac{1}{1-\alpha_n\gamma_n}\prod_{n=1}^{\infty}\frac{1}{(\alpha_n\gamma_n-1)^{\frac{1}{2}}}\\ &\quad\times \exp\left[\frac{1}{2}\sum\frac{[n]^2}{n}\frac{1}{1-\alpha_n\gamma_n}(\gamma_n(\beta_n^3)^2+2\beta_n^3\delta_n^3+\alpha_n(\delta_n^3)^2)\right],\end{aligned} \tag{4.2}$$

$$F_1(z_1,z_2) = \frac{1}{{}_B\langle r;0|0;r\rangle_B}\oint \mathrm{d}\omega_1\oint \mathrm{d}\omega\, \frac{(q^2-1)g_1}{q\omega^2(1-\omega_1q/\omega)(1-\omega q/\omega_1)(1-z_2q/\omega)}$$
$$\times \prod_{n=1}^{\infty}(-(\alpha_n\gamma_n-1)^{-1})\prod_{n=1}^{\infty}(\alpha_n\gamma_n-1)^{-\frac{1}{2}}\exp\left(\sum\frac{[n]^2}{n}\frac{1}{(\alpha_n\gamma_n-1)}\right.$$
$$\times\left\{(B_1-C_1)^2\frac{[2n]}{[n]}\frac{\gamma_n}{\alpha_n\gamma_n-1}+\gamma_n(B_1-C_1)\beta_n^2\right.$$

$$
\begin{aligned}
&- \gamma_n(B_1 - C_1)A_1 + (B_1 - C_1)\delta_n^2\bigg\}\bigg) \\
&\times \exp\bigg(\sum \frac{[n]^2}{n} \frac{1}{1-\alpha_n\gamma_n}\bigg\{\frac{1}{2}(\beta_n^3)^2 D\gamma_n + \frac{1}{2}\beta_n^3 D_1^2\gamma_n + \beta_n^3 D_1\gamma_n + \beta_n^3\delta_n^3 \\
&+ D_1\delta_n^3 + \tfrac{1}{2}\alpha_n(\delta_n^3)^2\bigg\}\bigg) \\
&+ \oint \mathrm{d}\omega_1 \oint \mathrm{d}\omega \, \frac{(q^2-1)g_1'}{q^2\omega z_2(1-\omega_1 q/\omega)(1-\omega q/\omega_1)(1-\omega/z_2q^3)} \\
&\times \prod_{n=1}^{\infty}(-(\alpha_n\gamma_n - 1)^{-1}) \prod_{n=1}^{\infty}(\alpha_n\gamma_n - 1)^{-\frac{1}{2}} \\
&\times \exp\bigg(\sum \frac{[n]^2}{n} \frac{1}{(\alpha_n\gamma_n - 1)}\bigg\{(B_1 - C_1)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n\gamma_n - 1} + \gamma_n(B_1 - C_1)\beta_n^2 \\
&- \gamma_n(B_1 - C_1)A_1 + (B_1 - C_1)\delta_n^2\bigg\}\bigg) \exp\bigg(\sum \frac{[n]^2}{n} \frac{1}{1-\alpha_n\gamma_n}\bigg\{\frac{1}{2}(\beta_n^3)^2 D_1'\gamma_n \\
&+ \frac{1}{2}\beta_n^3 (D_1')^2\gamma_n + \beta_n^3 D_1'\gamma_n + \beta_n^3\delta_n^3 + D_1'\delta_n^3 + \tfrac{1}{2}\alpha_n(\delta_n^3)^2\bigg\}\bigg), \qquad (4.3)
\end{aligned}
$$

where

$$
\begin{aligned}
g_1 = {} & \exp\bigg(-\sum \frac{q^{3n}z_2^{-n}\omega^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^n z_2^{-n}\omega^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^n z_2^n \omega^{-n}}{n}\bigg) \\
& \times \exp\bigg(-\sum \frac{q^{-n}z_2^{-n}\omega^n}{n}\bigg) \exp\bigg(\sum \frac{r^n z_2^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\bigg) \\
& \times \exp\bigg(\sum \frac{q^{4n}z_1^{-n}\omega_1^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^n\omega^{-n}\omega_1^n}{n}\bigg) \exp\bigg(-\sum \frac{q^{4n}\omega_1^{-2n}}{n}\bigg) \\
& \times \exp\bigg(-\sum \frac{r^n q^{2n}\omega_1^{-n}}{n}\bigg) \exp\bigg(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\bigg) \\
& \times \exp\bigg(-\sum \frac{q^{2n}z_2^n z_1^{-n}}{n}\bigg) \exp\bigg(\sum \frac{z_1^{-n}\omega_1^n}{n}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
g_1' = {} & \exp\bigg(\sum \frac{q^n z_2^n\omega^{-n}}{n}\bigg) \exp\bigg(-\sum \frac{q^{-n}z_2^{-n}\omega^n}{n}\bigg) \exp\bigg(\sum \frac{r^n z_2^{-n}}{n}\bigg) \\
& \times \exp\bigg(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^{4n}z_1^{-n}\omega_1^{-n}}{n}\bigg) \exp\bigg(\sum \frac{q^n\omega^{-n}\omega_1^n}{n}\bigg) \\
& \times \exp\bigg(-\sum \frac{q^{4n}\omega_1^{-2n}}{n}\bigg) \exp\bigg(-\sum \frac{r^n q^{2n}\omega_1^{-n}}{n}\bigg) \exp\bigg(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\bigg) \\
& \times \exp\bigg(-\sum \frac{q^{2n}z_2^n z_1^{-n}}{n}\bigg) \exp\bigg(\sum \frac{z_1^{-n}\omega_1^n}{n}\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 &= \sum \frac{q^{\frac{3}{2}n}z_1^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{1}{2}n}z_1^{-n}}{[n]} + \sum \frac{q^{\frac{1}{2}n}\omega^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n}\omega^{-n}}{[n]}, \\
B_1 &= \sum \frac{q^{\frac{5}{2}n}z_2^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{3}{2}n}z_2^{-n}}{[n]},
\end{aligned}
$$

$$D_1 = \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^n \omega^n}{[n]} + \alpha_n \sum \frac{q^{-n} \omega^{-n}}{[n]},$$

$$C_1 = \sum \frac{q^{\frac{1}{2}n} \omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n} \omega_1^{-n}}{[n]},$$

$$D_1' = \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^{-n} \omega^n}{[n]} + \alpha_n \sum \frac{q^n \omega^{-n}}{[n]},$$

$$\begin{aligned}
F_2(z_1, z_2) = {} & \frac{1}{{}_B\langle r; 0|0; r\rangle_B} \oint \mathrm{d}\omega \oint \mathrm{d}\omega_1 \, \frac{(1-q^{-2}) g_2}{z_1(1 - z_1 q^2/\omega)(1 - \omega/z_1)\omega_1 q(1 - z_2 q/\omega_1)} \\
& \times \prod_{n=1}^{\infty} (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
& \times \exp\left\{ \sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} \left[(B_2 - C_2)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_2 - C_2) \beta_n^2 \right.\right. \\
& \left.\left. - \gamma_n (B_2 - C_2) A_2 + (B_2 - C_2) \delta_n^2 \right]\right\} \exp\left\{ \sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \right. \\
& \left. \times \left[\frac{1}{2} (\beta_n^3)^2 D_2 \gamma_n + \frac{1}{2} \beta_n^3 D_2^2 \gamma_n + \beta_n^3 D_2 \gamma_n + \beta_n^3 \delta_n^3 + D_2 \delta_n^3 + \tfrac{1}{2} \alpha_n (\delta_n^3)^2 \right] \right\} \\
& + \oint \mathrm{d}\omega \oint \mathrm{d}\omega_1 \, \frac{(1-q^{-2}) g_2'}{z_1(1 - z_1 q^2/\omega)(1 - \omega/z_1) z_2 q^2 (1 - \omega_1 z_2 q^3)} \\
& \times \exp\left\{ \sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} \left[(B_2 - C_2)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_2 - C_2) \beta_n^2 \right.\right. \\
& \left.\left. - \gamma_n (B_2 - C_2) A_2 + (B_2 - C_2) \delta_n^2 \right]\right\} \\
& \times \prod_{n=1}^{\infty} (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \exp\left\{ \sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \right. \\
& \left. \times \left[\frac{1}{2} (\beta_n^3)^2 D_2' \gamma_n + \frac{1}{2} \beta_n^3 D_2'^2 \gamma_n + \beta_n^3 D_2' \gamma_n + \beta_n^3 \delta_n^3 + D_2' \delta_n^3 + \tfrac{1}{2} \alpha_n (\delta_n^3)^2 \right] \right\}, \qquad (4.4)
\end{aligned}$$

where

$$\begin{aligned}
g_2 = {} & \exp\left(\sum \frac{\omega^n z_1^{-n}}{n} \right) \exp\left(-\sum \frac{q^{2n} z_1^{-n} z_2^n}{n} \right) \exp\left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n} \right) \\
& \times \exp\left(-\sum \frac{q^{-n} \omega_1^n z_2^{-n}}{n} \right) \exp\left(\sum \frac{q^n \omega_1^{-n} z_2^n}{n} \right) \exp\left(\sum \frac{r^n z_2^{-n}}{n} \right) \\
& \times \exp\left(-\sum \frac{\omega^{-2n} q^{4n}}{n} \right) \exp\left(-\sum \frac{\omega^{-n} q^{2n} r^n}{n} \right) \exp\left(\sum \frac{q^{4n} \omega^{-n} z_1^{-n}}{n} \right) \\
& \times \exp\left(\sum \frac{q^{5n} \omega^{-n} \omega_1^{-n}}{n} \right) \exp\left(-\sum \frac{q^{3n} \omega_1^{-n} z_2^{-n}}{n} \right) \exp\left(\sum \frac{q^n \omega_1^{-n} z_2^{-n}}{n} \right) \\
& \times \exp\left(-\sum \frac{q^{2n} z_2^{-n} z_1^{-n}}{n} \right),
\end{aligned}$$

$$\begin{aligned}
g_2' = {} & \exp\left(\sum \frac{\omega^n z_1^{-n}}{n} \right) \exp\left(-\sum \frac{q^{2n} z_2^n z_1^{-n}}{n} \right) \exp\left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n} \right) \\
& \times \exp\left(-\sum \frac{q^{-n} \omega_1^n z_2^{-n}}{n} \right) \exp\left(\sum \frac{q^{3n} \omega_1^{-n} z_2^n}{n} \right) \exp\left(\sum \frac{r^n z_2^{-n}}{n} \right)
\end{aligned}$$

$$\times\ \exp\left(-\sum\frac{\omega^{-2n}q^{4n}}{n}\right)\exp\left(-\sum\frac{\omega^{-n}q^{2n}r^{n}}{n}\right)\exp\left(\sum\frac{q^{4n}\omega^{-n}z_1^{-n}}{n}\right)$$
$$\times\ \exp\left(\sum\frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\right)\exp\left(-\sum\frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\right),$$

and

$$A_2=\sum\frac{q^{\frac{3}{2}n}z_1^n}{[n]}-\alpha_n\sum\frac{q^{-\frac{1}{2}n}z_1^{-n}}{[n]}+\sum\frac{q^{\frac{1}{2}n}\omega_1^n}{[n]}-\alpha_n\sum\frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]},$$
$$B_2=\sum\frac{q^{\frac{5}{2}n}z_2^n}{[n]}-\alpha_n\sum\frac{q^{-\frac{3}{2}n}z_2^{-n}}{[n]},$$
$$D_2=\sum\frac{q^{2n}z_2^n}{[n]}-\alpha_n\sum\frac{q^{-2n}z_2^{-n}}{[n]}-\sum\frac{q^{n}\omega_1^n}{[n]}+\alpha_n\sum\frac{q^{-n}\omega_1^{-n}}{[n]},$$
$$C_2=\sum\frac{q^{\frac{1}{2}n}\omega_1^n}{[n]}-\alpha_n\sum\frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]},$$
$$D_2'=\sum\frac{q^{2n}z_2^n}{[n]}-\alpha_n\sum\frac{q^{-2n}z_2^{-n}}{[n]}-\sum\frac{q^{-n}\omega_1^n}{[n]}+\alpha_n\sum\frac{q^{n}\omega_1^{-n}}{[n]},$$

$$\begin{aligned}F_3(z_1,z_2)={}&\frac{-1}{{}_B\langle r;0|0;r\rangle_B}\oint\mathrm{d}\omega\oint\mathrm{d}\omega_1\,\frac{(1-q^{-2})g_3}{z_1(1-z_1q^2/\omega)(1-\omega/z_1)\omega_1\omega(1-\omega q/\omega_1)(1-\omega_1q/\omega)}\\
&\times\prod_{n=1}^{\infty}(-(\alpha_n\gamma_n-1)^{-1})\prod_{n=1}^{\infty}(\alpha_n\gamma_n-1)^{-\frac{1}{2}}\\
&\times\exp\left\{\sum\frac{[n]^2}{n}\frac{1}{\alpha_n\gamma_n-1}\left[(B_3-C_3)^2\frac{[2n]}{[n]}\frac{\gamma_n}{\alpha_n\gamma_n-1}+\gamma_n(B_3-C_3)\beta_n^2\right.\right.\\
&\left.\left.-\gamma_n(B_3-C_3)A_3+(B_3-C_3)\delta_n^2\right]\right\}\exp\left\{\sum\frac{[n]^2}{n}\frac{1}{1-\alpha_n\gamma_n}\right.\\
&\left.\times\left[\frac{1}{2}(\beta_n^3)^2D_3\gamma_n+\frac{1}{2}\beta_n^3D_3^2\gamma_n+\beta_n^3D_3\gamma_n+\beta_n^3\delta_n^3+D_3\delta_n^3+\tfrac{1}{2}\alpha_n(\delta_n^3)^2\right]\right\}\\
&+\oint\mathrm{d}\omega\oint\mathrm{d}\omega_1\,\frac{(1-q^{-2})g_3'}{z_1(1-z_1q^2/\omega)(1-\omega/z_1)\omega\omega_1(1-\omega q/\omega_1)(1-\omega_1q/\omega)}\\
&\times\prod_{n=1}^{\infty}(-(\alpha_n\gamma_n-1)^{-1})\prod_{n=1}^{\infty}(\alpha_n\gamma_n-1)^{-\frac{1}{2}}\\
&\times\exp\left\{\sum\frac{[n]^2}{n}\frac{1}{\alpha_n\gamma_n-1}\left[(B_3-C_3)^2\frac{[2n]}{[n]}\frac{\gamma_n}{\alpha_n\gamma_n-1}+\gamma_n(B_3-C_3)\beta_n^2\right.\right.\\
&\left.\left.-\gamma_n(B_3-C_3)A_3+(B_3-C_3)\delta_n^2\right]\right\}\exp\left\{\sum\frac{[n]^2}{n}\frac{1}{1-\alpha_n\gamma_n}\right.\\
&\left.\times\left[\frac{1}{2}(\beta_n^3)^2D_3'\gamma_n+\frac{1}{2}\beta_n^3D_3'^2\gamma_n+\beta_n^3D_3'\gamma_n+\beta_n^3\delta_n^3+D_3'\delta_n^3+\tfrac{1}{2}\alpha_n(\delta_n^3)^2\right]\right\},\end{aligned}\tag{4.5}$$

where

$$g_3=\exp\left(-\sum\frac{\omega^nz_1^{-n}}{n}\right)\left(-\sum\frac{q^{2n}z_1^{-n}z_2^n}{n}\right)\exp\left(\sum\frac{q^n\omega^{-n}\omega_1^n}{n}\right)$$
$$\times\ \exp\left(-\sum\frac{q^{3n}\omega_1^{-n}z_2^n}{n}\right)\exp\left(\sum\frac{q^n\omega_1^{-n}z_2^n}{n}\right)\exp\left(\sum\frac{r^nz_2^{-n}}{n}\right)$$

$$\times \exp\left(-\sum \frac{\omega^{-2n}q^{4n}}{n}\right)\exp\left(-\sum \frac{\omega^{-n}q^{2n}r^{n}}{n}\right)\exp\left(\sum \frac{q^{4n}\omega^{-n}z_1^{-n}}{n}\right)$$
$$\times \exp\left(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\right)\exp\left(-\sum \frac{q^{3n}\omega_1^{-n}z_2^{-n}}{n}\right)\exp\left(\sum \frac{q^{n}\omega_1^{-n}z_2^{-n}}{n}\right)$$
$$\times \exp\left(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\right),$$
$$g_3' = \exp\left(-\sum \frac{\omega^{n}z_1^{-n}}{n}\right)\exp\left(\sum \frac{q^{n}\omega^{-n}\omega_1^{n}}{n}\right)\left(-\sum \frac{q^{2n}z_1^{-n}z_2^{n}}{n}\right)$$
$$\times \exp\left(\sum \frac{r^{n}z_2^{-n}}{n}\right)\exp\left(-\sum \frac{\omega^{-2n}q^{4n}}{n}\right)\exp\left(-\sum \frac{\omega^{-n}q^{2n}r^{n}}{n}\right)$$
$$\times \exp\left(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\right)\exp\left(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\right)\exp\left(\sum \frac{q^{4n}\omega^{-n}z_1^{-n}}{n}\right),$$

and

$$A_3 = \sum \frac{q^{\frac{3}{2}n}z_1^{n}}{[n]} - \alpha_n \sum \frac{q^{-\frac{1}{2}n}z_1^{-n}}{[n]} + \sum \frac{q^{\frac{1}{2}n}\omega_1^{n}}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]},$$
$$B_3 = \sum \frac{q^{\frac{5}{2}n}z_2^{n}}{[n]} - \alpha_n \sum \frac{q^{-\frac{3}{2}n}z_2^{-n}}{[n]},$$
$$D_3 = \sum \frac{q^{2n}z_2^{n}}{[n]} - \alpha_n \sum \frac{q^{-2n}z_2^{-n}}{[n]} - \sum \frac{q^{n}\omega_1^{n}}{[n]} + \alpha_n \sum \frac{q^{-n}\omega_1^{-n}}{[n]},$$
$$C_3 = \sum \frac{q^{\frac{1}{2}n}\omega_1^{n}}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]},$$
$$D_3' = \sum \frac{q^{2n}z_2^{n}}{[n]} - \alpha_n \sum \frac{q^{-2n}z_2^{-n}}{[n]} - \sum \frac{q^{-n}\omega_1^{n}}{[n]} + \alpha_n \sum \frac{q^{n}\omega_1^{-n}}{[n]}.$$

We now derive the difference equations satisfied by the one-point functions. From (2.11) and (A.15), (A.16), one obtains

$$\Phi_i^*(z^{-1})|\alpha; r\rangle_B = \sum_j \bar{K}_i^{\,j}(zq)\Phi_j^*(zq^2)|\alpha; r\rangle_B, \tag{4.6}$$
$${}_B\langle r; \alpha|\Phi_i(z)(-1)^{[i]} = \sum_j {}_B\langle r; \alpha|\Phi_j(z^{-1}q^{-2})\bar{K}_i^{\,j}(zq)(-1)^{[j]}. \tag{4.7}$$

From (A.14), one derives the exchange relations

$$\Phi_i^*(z_1)\Phi_j(z_2) = \sum_{kl} \tilde{R}\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_l(z_2)\Phi_k^*(z_1)(-1)^{[k][l]}, \tag{4.8}$$

where $\tilde{R}(z) = R^{-1,st_1,-1}(z)$.

Using (3.2), (3.3), (4.6)–(4.8), (A.14) and (A.7)–(A.10), we get the difference equations

$$F_i(z_1q^{-2}, z_2) = \sum_{j,k,l,m,n} (-1)^{[k][l]+[i]+[j]+[n]} K_j^i(z_1q^{-2})\tilde{R}(z_1^{-1}z_2^{-1}q^2)_{ji}^{lk}$$
$$\times \bar{K}_l^m(z_1q^{-1})\bar{R}\left(\frac{z_1}{z_2}\right)_{m\ k}^{n\ n} F_n(z_1, z_2), \tag{4.9}$$
$$F_i(z_1, z_2q^2) = \sum_{j,k,l,m,n} (-1)^{[k][l]+[l]+[m]+[n]} K_i^{\,j}(z_2^{-1}q^{-2})\tilde{R}(z_1z_2q^2)_{kl}^{ij}$$
$$\times \bar{K}_l^m(z_2^{-1}q^{-1})\bar{R}\left(\frac{z_1}{z_2}\right)_{k\ m}^{n\ n} F_n(z_1, z_2). \tag{4.10}$$

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Appendix A

A.1. Bosonization of $U_q[\widehat{sl(2|1)}]$

In this appendix, we briefly review the bosonization of $U_q[\widehat{sl(2|1)}]$ at level one and the corresponding vertex operators [7, 25]. The Cartan matrix of $U_q[\widehat{sl(2|1)}]$ is

$$(a_{ij}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

where $i, j = 0, 1, 2$.

In terms of the Drinfeld generators: $\{d, X_m^{\pm,i}, h_n^i, (K^i)^{\pm 1}, \gamma^{\pm 1/2} | i = 1, 2,\ m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}\}$, the defining relations of $U_q[\widehat{sl(2|1)}]$ read

$$\begin{aligned}
&\gamma \text{ is central}, \qquad [K^i, h_m^j] = 0, \qquad [d, K^i] = 0, \qquad [d, h_m^j] = m h_m^j, \\
&[h_m^i, h_n^j] = \delta_{m+n,0} \frac{[a_{ij} m](\gamma^m - \gamma^{-m})}{m(q - q^{-1})}, \\
&K^i X_m^{\pm,j} = q^{\pm a_{ij}} X_m^{\pm,j} K^i, \qquad [d, X_m^{\pm,j}] = m X_m^{\pm,j}, \\
&[h_m^i, X_n^{\pm,j}] = \pm \frac{[a_{ij} m]}{m} \gamma^{\pm |m|/2} X_{n+m}^{\pm,j}, \\
&[X_m^{+,i}, X_n^{-,j}] = \frac{\delta_{i,j}}{q - q^{-1}} (\gamma^{(m-n)/2} \psi_{m+n}^{+,j} - \gamma^{-(m-n)/2} \psi_{m+n}^{-,j}), \\
&[X_m^{\pm,2}, X_n^{\pm,2}] = 0, \\
&[X_{m+1}^{\pm,i}, X_n^{\pm,j}]_{q^{\pm a_{ij}}} + [X_{n+1}^{\pm,j}, X_m^{\pm,i}]_{q^{\pm a_{ij}}} = 0, \qquad \text{for } a_{ij} \neq 0,
\end{aligned}$$

where $[m] = (q^m - q^{-m})/(q - q^{-1})$, $[X, Y]_\xi = XY - (-1)^{[X][Y]} \xi Y X$ and $[X, Y]_1 \equiv [X, Y]$; the $\mathbb{Z}_2$-grading of the Drinfeld generators is: $[X_m^{\pm,2}] = 1$ for $m \in \mathbb{Z}$ and zero otherwise.

Introduce the bosonic q-oscillators [25] $\{a_n^1, a_n^2, b_n, c_n, Q_{a^1}, Q_{a^2}, Q_b, Q_c | n \in \mathbb{Z}\}$, which satisfy the commutation relations

$$\begin{aligned}
&[a_m^i, a_n^j] = \delta_{i,j} \delta_{m+n,0} \frac{[m]^2}{m}, \qquad [a_0^i, Q_{a^j}] = \delta_{i,j}, \\
&[b_m, b_n] = -\delta_{m+n,0} \frac{[m]^2}{m}, \qquad [b_0, Q_b] = -1, \\
&[c_m, c_n] = \delta_{m+n,0} \frac{[m]^2}{m}, \qquad [c_0, Q_c] = 1.
\end{aligned}$$

Define the generating functions for the Drinfeld basis as $X_i^{\pm}(z) = \sum_{m \in \mathbb{Z}} X_m^{\pm,i} z^{-m-1}$, and introduce h_0^i by setting $K^i = q^{h_0^i}$. Define $Q_{h^1} = Q_{a^1} - Q_{a^2}$, $Q_{h^2} = Q_{a^2} + Q_b$ and $h_i(z; \beta)$ by

$$h_i(z; \beta) = -\sum_{n \neq 0} \frac{h_n^i}{[n]} q^{-\beta |n|} z^{-n} + Q_{h^i} + h_0^i \ln z, \tag{A.1}$$

where β is a parameter. Other bosonic fields are defined similarly.

The Drinfeld generators at level one are realized by the free-boson fields as [25]

$$h_m^1 = a_m^1 q^{-|m|/2} - a_m^2 q^{|m|/2}, \qquad h_m^2 = a_m^2 q^{-|m|/2} + b_m q^{-|m|/2}, \qquad m \in \mathbb{Z},$$
$$X_1^{\pm}(z) = \pm : \mathrm{e}^{\pm h_1(z;\pm\frac{1}{2})} : \mathrm{e}^{\pm \mathrm{i}\pi a_0^1}, \qquad X_2^{+}(z) = \,: \mathrm{e}^{h_2(z;\frac{1}{2})} \mathrm{e}^{c(z;0)} : \mathrm{e}^{-\mathrm{i}\pi a_0^1},$$
$$X_2^{-}(z) = \,: \mathrm{e}^{-h_2(z;-\frac{1}{2})} [\partial_z \mathrm{e}^{-c(z;0)}] : \mathrm{e}^{\mathrm{i}\pi a_0^1}, \qquad \gamma = q,$$

where

$$\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} \; : O :$$

stands for the usual normal ordering of O.

Consider the bosonic Fock spaces $F_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$, generated by a_{-m}^i, b_{-m}, c_{-m} $(m > 0)$ over the vacuum vectors $|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle$:

$$F_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = \boldsymbol{C}[a_{-1}^i, a_{-2}^i, \ldots; b_{-1}, \ldots; c_{-1}, \ldots]|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle, \tag{A.2}$$

where

$$a_m^i|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0, \qquad b_m|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0,$$
$$c_m|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0, \qquad \text{for } m > 0,$$
$$|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = \mathrm{e}^{\lambda_1 Q_{a^1} + \lambda_2 Q_{a^2} + \lambda_3 Q_b + \lambda_4 Q_c}|0, 0, 0, 0\rangle.$$

Introduce the following spaces:

$$F_{(\alpha;\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+i,\beta-i+j,\beta-\alpha+j,-\alpha+j}. \tag{A.3}$$

It can be shown that the bosonized action of $U_q[\widehat{sl(2|1)}]$ on $F_{(\alpha;\beta)}$ is closed. To obtain the irreducible subspaces in $F_{(\alpha;\beta)}$, it convenient to introduce a pair of fermionic currents [25,26]:

$$\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n-1} = \,: \mathrm{e}^{c(z;0)} :, \qquad \xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n} = \,: \mathrm{e}^{-c(z;0)} :,$$

The mode expansion of $\eta(z), \xi(z)$ is well defined on $F_{(\alpha;\beta)}$ for $\alpha \in \mathbb{Z}$, and it satisfies the following relations:

$$\xi_m \xi_n + \xi_n \xi_m = \eta_m \eta_n + \eta_n \eta_m = 0, \qquad \xi_m \eta_n + \eta_n \xi_m = \delta_{m,n}.$$

Since η_0 commutes (or anticommutes) with $U_q[\widehat{sl(2|1)}]$, η_0 plays the role of screening charge and $\eta_0\xi_0$ qualifies as the projector from $F_{(\alpha;\beta)}$ to the kernel of η_0. Set $\lambda_\alpha = (1-\alpha)\Lambda_0 + \alpha\Lambda_2$, $\alpha \in \mathbb{Z}$, where Λ_i $(i = 0, 1, 2)$ are the fundamental weights of $U_q[\widehat{sl(2|1)}]$, and

$$\mu_\alpha = \begin{cases} \Lambda_\alpha, & \alpha = 0, 1, 2 \\ \lambda_{\alpha-1} & \text{for } \alpha > 2 \\ \lambda_\alpha & \text{for } \alpha < 0. \end{cases} \tag{A.4}$$

Define $V(\mu_\alpha) = \eta_0\xi_0 F_{(\alpha,\beta-\alpha)}$. Following [7,25], $V(\mu_\alpha)$ $(\alpha \in \mathbb{Z})$ are the irreducible highest-weight $U_q[\widehat{sl(2|1)}]$-modules with the highest weight μ_α.

A.2. Level-one vertex operators

Let $V(\lambda)$ be the highest-weight $U_q[\widehat{sl(2|1)}]$-module with the highest weight λ. Consider the following intertwiners of $U_q[\widehat{sl(2|1)}]$-modules:

$$\Phi_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z, \qquad \Phi_\lambda^{\mu V^*}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{*S}.$$

They are intertwiners in the sense that for any $x \in U_q[\widehat{sl(2|1)}]$,

$$\Theta(z)x = \Delta(x)\Theta(z), \qquad \Theta(z) = \Phi(z), \Phi^*(z), \tag{A.5}$$

the grading of these operators is: $[\Theta(z)] = 0$. $\Phi(z)$ is called a type I (dual) vertex operator [9]. We expand the vertex operator as

$$\Phi(z) = \sum_{j=1,2,3} \Phi(z)_j \otimes v_j, \qquad \Phi^*(z) = \sum_{j=1,2,3} \Phi^*(z)_j \otimes v_j^{*S}. \tag{A.6}$$

Define the operators $\phi_j(z)$, $\phi_j^*(z)$, $\psi_j(z)$ and $\psi_j^*(z)$ $(j = 1, 2, 3)$ by

$$\phi_3(z) = \ : \mathrm{e}^{-h_2^*(q^2z;-\frac{1}{2})+c(q^2z;0)} : \mathrm{e}^{-\mathrm{i}\pi a_0^2}, \tag{A.7}$$

$$\phi_2(z) = -[\phi_3(z), X_0^{-,2}]_{q^{-1}}, \qquad \phi_1(z) = [\phi_2(z), X_0^{-,1}]_q, \tag{A.8}$$

$$\phi_1^*(z) = \ : \mathrm{e}^{h_1^*(qz;-\frac{1}{2})} : \mathrm{e}^{\mathrm{i}\pi a_0^2}, \tag{A.9}$$

$$\phi_2^*(z) = -q^{-1}[\phi_1^*(z), X_0^{-,1}]_q, \qquad \phi_3^*(z) = q^{-1}[\phi_2^*(z), X_0^{-,2}]_q, \tag{A.10}$$

where $h_m^{*1} = -h_m^2$, $h_m^{*2} = -h_m^1 - ([2m]/[m])h_m^2$ and $Q_{h^{*1}} = -Q_{h^2}$, $Q_{h^{*2}} = -Q_{h^1} - 2Q_{h^2}$. Since the operators $\phi_i(z)$, $\phi_i^*(z)$ commute (or anticommute) with η_0, we define

$$\Phi_i(z) = \eta_0\xi_0\phi_i(z)\eta_0\xi_0, \qquad \Phi_i^*(z) = \eta_0\xi_0\phi_i^*(z)\eta_0\xi_0. \tag{A.11}$$

According [7, 25], the vertex operators $\Phi(z)$ and $\Phi^*(z)$, equation (A.6), given by (A.11) are the only type I vertex operators of $U_q[\widehat{sl(2|1)}]$ which intertwine the level-one irreducible highest-weight $U_q[\widehat{sl(2|1)}]$-modules $V(\mu_\alpha)$ $(\alpha \in \mathbb{Z})$:

$$\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_z, \qquad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V_z^{*s}.$$

It is shown [7] that the above vertex operators satisfy the graded Faddeev–Zamolodchikov algebra

$$\Phi_j(z_2)\Phi_i(z_1) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_k(z_1)\Phi_l(z_2)(-1)^{[i][j]}, \tag{A.12}$$

$$\Phi_j^*(z_2)\Phi_i^*(z_1) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{kl}^{ij} \Phi_k^*(z_1)\Phi_l^*(z_2)(-1)^{[i][j]}, \tag{A.13}$$

$$\Phi_j(z_2)\Phi_i^*(z_1) = \sum_{kl} \bar{R}\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_k^*(z_1)\Phi_l(z_2)(-1)^{[k][l]}, \tag{A.14}$$

where $\bar{R}(z) = R^{-1,st_1}(z)$. Moreover, the vertex operators have the following invertibility relations:

$$\Phi_i(z)\Phi_j^*|_{V(\Lambda_\alpha)} = -(-1)^{[j]}\delta_{ij}\,\mathrm{id}_{V(\Lambda_\alpha)}, \qquad -\sum_k(-1)^{[k]}\Phi_k^*(z)\Phi_k(z)|_{V(\Lambda_\alpha)} = \mathrm{id}|_{V(\Lambda_\alpha)}, \tag{A.15}$$

$$\Phi_i^*(zq^2)\Phi_j(z)|_{V(\Lambda_\alpha)} = \delta_{ij}q^{2\rho_i}\,\mathrm{id}|_{V(\Lambda_\alpha)}, \qquad \sum_k q^{-2\rho_k}\Phi_k(z)\Phi_k^*(zq^2)|_{V(\Lambda_\alpha)} = \mathrm{id}|_{V(\Lambda_\alpha)}. \tag{A.16}$$

Appendix B

In this appendix, we give the normal ordering relations of fundamental bosonic fields:

$$\mathrm{e}^{h_1(z_1;\beta_1)}\mathrm{e}^{h_1(z_2;\beta_2)} = (z_1 - q^{-(\beta_1+\beta_2)+1}z_2)(z_1 - q^{-(\beta_1+\beta_2)-1}z_2) : \mathrm{e}^{h_1(z_1;\beta_1)}\mathrm{e}^{h_1(z_2;\beta_2)} :,$$

$$\mathrm{e}^{h_1(z_1;\beta_1)}\mathrm{e}^{h_2(z_2;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : \mathrm{e}^{h_1(z_1;\beta_1)}\mathrm{e}^{h_2(z_2;\beta_2)} :,$$

$$e^{h_2(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_2(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} :,$$
$$e^{h_2(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} = : e^{h_2(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} :,$$
$$e^{h_i(z_1;\beta_1)}e^{h_j^*(z_2;\beta_2)} = (z_1 - q^{-(\beta_1+\beta_2)}z_2)^{\delta_{ij}} : e^{h_i(z_1;\beta_1)}e^{h_j^*(z_2;\beta_2)} :,$$
$$e^{h_i^*(z_1;\beta_1)}e^{h_j(z_2;\beta_2)} = (z_1 - q^{-(\beta_1+\beta_2)}z_2)^{\delta_{ij}} : e^{h_i^*(z_1;\beta_1)}e^{h_j(z_2;\beta_2)} :,$$
$$e^{h_1^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} = : e^{h_1^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} :,$$
$$e^{h_1^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_1^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} :,$$
$$e^{h_2^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_2^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} :,$$
$$e^{h_2^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} = \frac{1}{(z_1 - q^{-(\beta_1+\beta_2)+1}z_2)(z_1 - q^{-(\beta_1+\beta_2)-1}z_2)} : e^{h_2^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} :,$$
$$e^{c(z_1;\beta_1)}e^{c(z_2;\beta_2)} = (z_1 - q^{-(\beta_1+\beta_2)}z_2) : e^{c(z_1;\beta_1)}e^{c(z_2;\beta_2)} : .$$

Appendix C

We here summarize the formulae concerning coherent states of bosons which have been used in section 4.

The coherent states $|\zeta^1, \zeta^2, \zeta^3\rangle$ and $\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|$ in the Fock space $F_{(0;\beta)}$ and its dual space $F^*_{(0;\beta)}$ are defined by

$$|\zeta^1, \zeta^2, \zeta^3\rangle = \exp\left\{\sum_{m=1}\sum_{i=1}^{2}\frac{m}{[m]^2}\zeta_m^i h_{-m}^{*i} + \sum_{m=1}\frac{m}{[m]^2}\zeta_m^3 c_{-m}\right\}|\beta, \beta, \beta, 0\rangle, \tag{C.1}$$

$$\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3| = \langle\beta, \beta, \beta, 0|\exp\left\{\sum_{m=1}\sum_{i=1}^{2}\frac{m}{[m]^2}\bar{\zeta}_m^i h_m^{*i} + \sum_{m=1}\frac{m}{[m]^2}\bar{\zeta}^3 c_m\right\} \tag{C.2}$$

where ζ_m^l and $\bar{\zeta}_m^l$ $(l = 1, 2, 3; m = 1, 2, \ldots)$ are complex conjugate parameters.

Noting that

$$h_m^i|\beta, \beta, \beta, 0\rangle = 0, \qquad \langle\beta, \beta, \beta, 0|h_{-m}^i = 0, \qquad i = 1, 2,\ m \geqslant 1,$$
$$c_m|\beta, \beta, \beta, 0\rangle = 0, \qquad \langle\beta, \beta, \beta, 0|c_{-m} = 0, \qquad m \geqslant 1,$$

one can easily verify that

$$h_m^i|\zeta^1, \zeta^2, \zeta^3\rangle = \zeta_m^i|\zeta^1, \zeta^2, \zeta^3\rangle, \qquad \langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|h_{-m}^i = \bar{\zeta}_m^i\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|, \qquad i = 1, 2,$$
$$c_m|\zeta^1, \zeta^2, \zeta^3\rangle = \zeta_m^3|\zeta^1, \zeta^2, \zeta^3\rangle, \qquad \langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|c_{-m} = \bar{\zeta}_m^3\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|.$$

One can also show that the coherent states $\{|\zeta^1, \zeta^2, \zeta^3\rangle\}$ $(\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|\})$ form a complete basis in Fock space $F_{(0;\beta)})$ $(F^*_{(0;\beta)})$. That is, one can verify the completeness relation

$$\mathrm{id}_{F_{(0;\beta)}} = \int\prod_{m=1}^{\infty}\frac{d\zeta_m^1\, d\bar{\zeta}_m^1\, d\zeta_m^2\, d\bar{\zeta}_m^2\, d\zeta_m^3\, d\bar{\zeta}_m^3}{([m]^2/m)\det([a_{ij}m][m]/m)}\exp\left\{-\sum_{m=1}^{\infty}\sum_{i,j=1}^{2}\frac{K_{ij}(m)m}{[m]}\zeta_m^i\bar{\zeta}_m^j\right\}$$
$$\times\, |\zeta^1, \zeta^2, \zeta^3\rangle\langle\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|, \tag{C.3}$$

where $K_{ij}(n)$ is a 2×2 matrix satisfying

$$\sum_{l=1}^{2}K_{il}(n)[a_{lj}n] = \delta_{ij}.$$

One may also derive the following identity:

$$\int \prod_{m=1}^{\infty} \frac{\mathrm{d}\zeta_m^1\,\mathrm{d}\bar{\zeta}_m^1\,\mathrm{d}\zeta_m^2\,\mathrm{d}\bar{\zeta}_m^2\,\mathrm{d}\zeta_m^3\,\mathrm{d}\bar{\zeta}_m^3}{([m]^2/m)\det([a_{ij}m][m]/m)} \exp\left\{-\frac{1}{2}\sum_{m=1}^{\infty}\lambda_m(\bar{\zeta}_m^1, \bar{\zeta}_m^2, \bar{\zeta}_m^3, \zeta_m^1, \zeta_m^2, \zeta_m^3)\mathcal{A}_m \begin{pmatrix} \bar{\zeta}_m^1 \\ \bar{\zeta}_m^2 \\ \bar{\zeta}_m^3 \\ \zeta_m^1 \\ \zeta_m^2 \\ \zeta_m^3 \end{pmatrix} \right.$$
$$\left. + \sum_{m=1}^{\infty}(\bar{\zeta}_m^1, \bar{\zeta}_m^2, \bar{\zeta}_m^3, \zeta_m^1, \zeta_m^2, \zeta_m^3)\mathcal{B}_m \right\}$$
$$= \prod_{m=1}^{\infty}(-\det\mathcal{A}_m)^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\sum_{m=1}^{\infty}\frac{[m]^2}{m}\det\left(\frac{[a_{ij}m][m]}{m}\right)\mathcal{B}_m^t\mathcal{A}_m^{-1}\mathcal{B}_m\right\}, \tag{C.4}$$

where $\mathcal{A}_m$ are invertible constant 6×6 matrices and $\mathcal{B}_m$ are constant 6-component vectors.

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